WHY CH IS FALSE

SOME UNINTUITIVE CONSEQUENCES

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Section 1. *Background*

§ 1 A. Philosophy

Let me start off by saying that the continuum hypothesis (CH) is *independent* of the generally agreed upon assumptions of ZFC. What this means is that the axioms of ZFC do not determine whether CH holds in the same way the group axioms do not determine commutativity. Contained in this is the assumption that ZFC is consistent, something which is also not provable from ZFC. In this way, ZFC is not the end-all-be-all of what can be understood or known. Thinking about these independent propositions is then similar to thinking as a scientist. Science does not have direct access to how the world works: they must conjecture, and come up with theories to explain the facts. We too do not have access to the truth or falsity of such propositions, so we must look at the surrounding evidence, conjecture, and form beliefs as a result.

What is presented here is not a proof that CH is false, but instead some counter-intuitive consequences that make some believe that CH is false. Of course, what is unintuitive for one may be intuitive for another. There are also a whole host of more complicated set theoretic reasons some have for thinking CH is false, and in particular for thinking that $|\mathbb{R}| = \aleph_2$ (cf. [4]). Such reasons cannot be presented in a meaningful way in an hour, and I admit that I don't think they are all that convincing to non-set theorists[i].

There are other reasons for thinking that CH is false that are not motivated by results of CH, but instead by unintuitive propositions which imply CH. In particular, there is the notion of Gödel's constructible universe, L. As the name suggests, this universe is constructed level by level by, instead of the full powerset, taking definable subsets. While the set theoretic universe is form by the recursion, taking γ as a limit ordinal,

$$\mathrm{V}_0 = \emptyset, \qquad \mathrm{V}_{\alpha+1} = \mathcal{P}(\mathrm{V}_\alpha), \qquad \mathrm{V}_\gamma = \bigcup_{\alpha<\gamma} \mathrm{V}_\alpha,$$

L is formed by having $\mathrm{L}_{\alpha+1} = \{X \subseteq \mathrm{L}_\alpha \colon X \text{ is definable over } \mathrm{L}_\alpha\}$. The resulting hierarchy of constructible sets L satisfies CH, and in fact the *generalized* continuum hypothesis (GCH), which states that the cardinal after κ is always 2^κ.

[i] The set theory class in the 2019 Spring semester spent at least half of the course proving that $|\mathbb{R}| = \aleph_2$ from some unrelated forcing axioms, for example. The actual fact of whether such axioms apply is provably unprovably consistent relative to the consistency ZFC.

The assumption that the universe of sets V = L is unpopular, and seemingly wrong by the principle of maximization: L is the smallest inner model of ZFC[ii], so why should the universe of all abstract objects consist merely of what is necessary from the axioms? It would seem that the actual set theoretic universe couldn't be constructed merely by formulas in this way. One way to easily reject V = L is then to reject one of its consequences, like CH.

There are other heuristics for thinking that CH—and more generally GCH—is false. There are theorems due to forcing that show that $|\mathbb{R}|$ can be *any* cardinal of uncountable cofinality[iii]. It would seem strange for $|\mathbb{R}|$ to be as small as it possibly could. There are other theorems, which relate $|\mathbb{R}|$ to the cardinality of other things. For example, $|\mathbb{R}|$ is the cardinality of countable, linear orders up to isomorphism. $\aleph_1$ is the number of countable ordinals. It would seem strange for the number of countable linear orders, of which there are many strange and complicated structures, to be the same as the number of countable well-orders, which are all very neatly organized and easy to describe.

I'm putting these more heuristic, non-mathematical arguments here at the beginning just to cover my bases, and note that there are a lot of things not covered in this document. I want the rest of the document to focus on some fun, unintuitive, easy-to-prove results of CH in fields other than set theory rather than the minutia of how to think about independence results. Most of these are classic results found in [3].

§ 1 B. An introduction to transfinite recursion

As CH is a statement about the ordinal ω_1, it will be useful to review transfinite recursion and transfinite induction. Ordinals are originally understood as well-orders, and are initial segments of each other. In this way, we have the defining property that for any set of ordinals, there is a least member[iv]. This defining property is useful mostly because it allows us to conclude the following result.

1 B • 1. Theorem

For any ordinal α, one of the following holds:
- $\alpha = 0$;
- $\alpha = \beta + 1$ for some ordinal β, meaning α is a *successor ordinal*; or
- $\alpha = \sup_{\beta<\alpha} \beta$, meaning α is a *limit ordinal*.

And this in turn allows us to use this characterization to come up with constructions stage by stage just by specifying what happens at each stage: at successor stages, and at limit stages. This idea is called *transfinite recursion*, and is a theorem of ZFC stated in the clumsy, proper way.

Transfinite recursion can be used for all sorts of constructions. For example, the V_αs just mentioned above. Taking $V_0 = \emptyset$, and $V_{\alpha+1} = \mathcal{P}(V_\alpha)$ with limits as unions defines the entire universe of sets: $V = \bigcup_{\alpha\in\mathrm{Ord}} V_\alpha$. For another example, we can grow the tree of binary sequences of length ω_1.

Let $T_0 = \{\Lambda\}$ where Λ is the empty-sequence[v]. For T_α already defined, let $T_{\alpha+1}$ be the set of all elements of T_α extended by 0, and extended by 1:

$$T_{\alpha+1} = T_\alpha \cup \{f^\frown 0 : f \in T_\alpha\} \cup \{f^\frown 1 : f \in T_\alpha\},$$

where $x^\frown y$ denotes concatenation. For limit ordinals α, define $T_\alpha = \bigcup_{\beta<\alpha} T_\beta$. The resulting tree T_{ω_1} is the binary tree of height ω_1 when the elements are ordered by end-extension.

For a final example, we can talk about the sequence of $\aleph$s or ωs. Starting with $\omega_0 = \omega$, we can define by transfinite

[ii] By an *inner model*, I just mean a transitive model which contains all the ordinals of V. It being the smallest is a result of the absoluteness of its construction: any transitive inner model will necessarily contain L. A more general result is that of *condensation*, saying that even in smaller models, you must contain L up to whatever ordinal is your class of ordinals.

[iii] This just means that we can't write $|\mathbb{R}|$ as the countable union of subsets of size $< |\mathbb{R}|$, something trivially true under ZFC + CH, but provable from ZFC alone as well. And in fact, this one provable barrier is the *only* barrier for what $|\mathbb{R}|$ could be.

[iv] The ordering of the ordinals is $\in$, set membership.

[v] In traditional set theoretic interpretation, Λ is just the empty set.

recursion, for γ a limit,

$$\omega_{\alpha+1} := \text{ the least ordinal of cardinalty } > \omega_\alpha$$
$$\omega_\gamma = \bigcup_{\alpha<\gamma} \omega_\alpha.$$

In the context of cardinality, $|\omega_\alpha| = \aleph_\alpha$.

Section 2. Easier Results

§ 2 A. Covering real numbers

Formulated as $|\mathbb{R}| = \aleph_1$, CH is a statement about real numbers, and so it's appropriate that our results will be about real numbers. Often such results can really be considered properties of $\aleph_1$ formulated as results about $\mathbb{R}$. For example, the next result is like this.

2 A • 1. Result

(ZFC + CH) The plane $\mathbb{R}^2$ is the countable union of functions from the x-axis to the y-axis, and functions from the y-axis to the x-axis.

Proof .:.

Since $|\mathbb{R}| = \aleph_1$, by doing the result on $\aleph_1$, we can just apply a bijection to get the result for $\mathbb{R}$. Now each ordinal $\alpha < \omega_1$ is countable, and so we get a surjection $f_\alpha : \omega \to \alpha$. But then the map sending α to $f_\alpha(n)$ for a fixed n is a perfectly legitimate map from ω_1 to ω_1. Yet there are only countably many such maps: $F_n(\alpha) := f_\alpha(n)$ for $n \in \omega$. Moreover, each $\langle \alpha, \beta \rangle$ for $\beta < \alpha$ is the result of some $F_n(\alpha)$, since each f_α was a surjection onto α: $f_\alpha(n) = \beta$ for some n, and so $F_n(\alpha) = \beta$. Therefore, we have covered the lower triangle of the plane.

To get the upper triangle, we just do the same process for the y-axis, and thus get a covering of $\aleph_1 \times \aleph_1$ by $2 \cdot \aleph_0 = \aleph_0$ functions. So by applying the bijection with $\mathbb{R}$, we get the result for $\mathbb{R}$. ⊣

2 A • 2. Result

(ZFC + CH) $\mathbb{R}$ can be colored by countably many colors such that there are no $a, b, c, d \in \mathbb{R}$ of the same color such that $a + b = c + d$ (where a, b, c, and d are all distinct).

Proof .:.

Proceed by transfinite recursion of length ω_1 to color all points of $\mathbb{R}$. By CH, let $\{r_\alpha : \alpha < \omega_1\}$ enumerate the elements of $\mathbb{R}$. Note that the closure of a countable set under countably many operations is still countable. In particular, each set $X_\alpha = \{r_\beta : \beta < \alpha\}$ for $\alpha < \omega_1$ is countable. So if we take the closure of X_α under addition and subtraction, we get a set $R_\alpha \subseteq \mathbb{R}$ which is still countable, and we still have the property that $\mathbb{R} = \bigcup_{\alpha<\omega_1} R_\alpha$.

Define a coloring by transfinite recursion. Note that $R_0 = \emptyset$ already has its elements colored. At limit stages, $R_\alpha = \bigcup_{\beta<\alpha} R_\beta$, so at this stage, the elements of R_α are already colored. For successors, color each element of $R_{\alpha+1} \setminus R_\alpha$ a color different from the rest of $R_{\alpha+1} \setminus R_\alpha$ from a fixed countable rainbow. Since $R_{\alpha+1} \setminus R_\alpha$ is countable, we can do this.

Note that for $x = a + b - c \in R_{\alpha+1}$, we can't have all the $a, b, c \in R_\alpha$, since this would imply $x \in R_\alpha$ by closure under addition and subtraction. Thus there would need to be, say, $a \in R_{\alpha+1}$ which would then get a different color from x.

Because we're using colors from a set countable rainbow, we aren't ever introducing colors, and the result is that $\mathbb{R}$ is colored in a way that satisfies the statement. $\dashv$

The above result isn't too unintuitive in general, since we could just color each point of $\mathbb{R}$ a different color. The key thing that is surprising is that we're using only countably many colors. This same process can be generalized to other kinds of "colorings", though we might have to be more careful than just assigning colors arbitrarily.

2 A • 3. Result

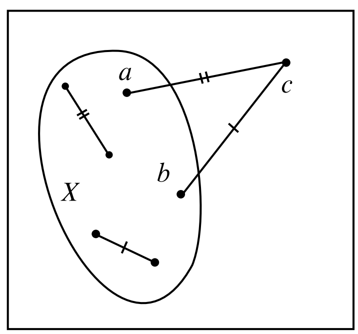
(ZFC + CH) There is a coloring of $\mathbb{R}^2$ with countably many colors, such that there are no $a, b, c, d \in \mathbb{R}^2$ of the same color with $\text{dist}(a, b) = \text{dist}(c, d)$ (where a, b, c, and d are all distinct).

Proof .:.

Enumerate $\mathbb{R}^2 = \{r_\alpha : \alpha < \omega_1\}$. For $A \subseteq \mathbb{R}^2$, let

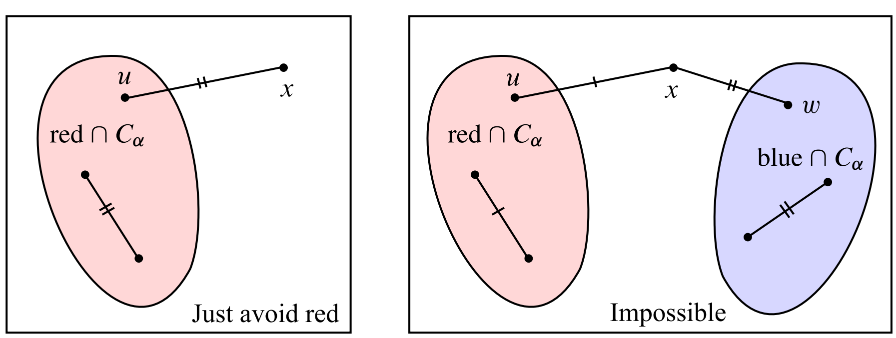
$$D(A) := \{\text{dist}(a, b) \in \mathbb{R} : a, b \in A\},$$

the set of distances between points of A. For each countable set $X \subseteq \mathbb{R}^2$, we can take the closure C under triangles with distances within X. This means that if $a, b \in C$, and $\text{dist}(a, c), \text{dist}(b, c) \in D(C)$, then $c \in C$. This closure, denoted $C(X)$, will still be countable. This can be visualized below, where $c \in C(X) \setminus X$.



Let $X_\alpha = \{r_\beta : \beta < \alpha\}$ for $\alpha < \omega_1$. From this sequence, define the sequence of $\langle C_\alpha : \alpha < \omega_1 \rangle$ by $C_\alpha = C(X_\alpha)$. We thus get that $\mathbb{R} = \bigcup_{\alpha<\omega_1} C_\alpha$, and each C_α is countable. We will color according to the differences $C_{\alpha+1} \setminus C_\alpha$. For α a limit, $C_\alpha = \bigcup_{\beta<\alpha} C_\beta$ is already colored. $C_0 = \emptyset$ is also already colored, so we only need to deal with the successor case.

For $\alpha + 1$, for each of the countably many elements of $C_{\alpha+1} \setminus C_\alpha$, give each a distinct color from our fixed, countable rainbow. We must be careful about how we do this, though, because we might accidentally color a point the same distance from some others previously colored, as below.



To avoid this, note that for each $x \in C_{\alpha+1} \setminus C_\alpha$, there can't be two points in C_α forming with distances to x already found in $D(C_\alpha)$ just by construction of C_α. But there might be a color red where an element already colored, $u \in C_\alpha$, whose distance to x can already be found in red $\cap C_\alpha$, meaning $\text{dist}(x, u) \in D(\text{red} \cap C_\alpha)$. In this case, we shouldn't color x red. Note that there can be only one color like this, though, since the existence of two such colors would imply $x \in C_\alpha$ by construction: $\text{dist}(x, u), \text{dist}(x, w) \in D(C_\alpha)$ implies $x \in C_\alpha$, as above.

So write $C_{\alpha+1} \setminus C_\alpha = \{x_n : n < \omega\}$, and for each x_n, let c_n be the corresponding color that needs to be avoided, and then set the color of x_n to be c_{n+1}. We also need to establish that all the c_ns are distinct, but this is easy enough, since if the color x_n needs to avoid has already been listed, then set c_n as any arbitrary color not in $\{c_i : i < n\}$. ⊣

2 A • 4. Result

(ZFC + CH) The plane $\mathbb{R}^2$ can be colored with two colors such that every horizontal line contains only countably many red points, and every vertical line contains only countably many blue points.

Proof .:.

By CH, enumerate $\mathbb{R} = \{r_\alpha : \alpha < \omega_1\}$. Set

$$\text{Red} = \{\langle r_\alpha, r_\beta\rangle : \alpha < \beta < \omega_1\},$$
$$\text{Blue} = \{\langle r_\alpha, r_\beta\rangle : \beta \le \alpha < \omega_1\}.$$

Consider the horizontal line $H = \mathbb{R} \times \{r_\beta\} = \{\langle x, r_\beta\rangle : x \in \mathbb{R}\}$. For $\langle x, r_\beta\rangle \in H \cap \text{Red}$, we must have that $x = r_\alpha$ for some $\alpha < \beta$, which means that $H \cap \text{Red}$ is countable, since $\beta < \omega_1$. Similarly, for the vertical line $V = \{r_\alpha\} \times \mathbb{R} = \{\langle r_\alpha, y\rangle : y \in \mathbb{R}\}$, $\langle r_\alpha, y\rangle \in V \cap \text{Blue}$ iff $y = r_\beta$ for some $\beta \le \alpha$. But again, that's only countably many possible reals. Hence $V \cap \text{Blue}$ is countable too. ⊣

Another interesting fact is that all three of the above results Result 2 A • 1, Result 2 A • 2, and Result 2 A • 4 are individually equivalent to CH. In fact, we can generalize Result 2 A • 4 to various bounds on the continuum [2].

§ 2 B. Weird sets of reals

Denote lebesgue measure by μ, calling μ-measure 0 sets just *measure 0*. The following set of reals is called a *sierpiński* set after Wacław Sierpiński, who showed this result. The existence of a sierpiński set does not require CH, but if sets of size $< |\mathbb{R}|$ are measure 0, then its existence implies CH.

2 B • 1. Result

(ZFC + CH) There is an uncountable subset $A \subseteq \mathbb{R}$ where every μ-measure 0 subset $B \subseteq A$ is countable.

Proof .:.

Every set of measure 0 is included in a G_δ set (a countable intersection of open sets) of measure 0, and there are $|\mathbb{R}|$ G_δ sets. By CH, we can enumerate the measure 0 sets in G_δ by $\{X_\alpha : \alpha < \omega_1\}$. Note that as a countable union, for each $\alpha < \omega_1$, $\bigcup_{\beta<\alpha} X_\beta$ is a set of measure zero.

So for each $\alpha < \omega_1$, let $y_\alpha \in \mathbb{R} \setminus \bigcup_{\beta<\alpha} X_\beta$. Now consider $Y = \{y_\alpha : \alpha < \omega_1\}$ which is uncountable of cardinality $\aleph_1$. Each measure 0 set $Z \subseteq \mathbb{R}$ is contained in some X_α for $\alpha < \omega_1$, and hence $Y \cap Z \subseteq Y \cap X_\gamma \subseteq \{y_\beta : \beta \le \alpha\}$ is countable. ⊣

2 B • 2. Result

(ZFC + CH) There is an uncountable $A \subseteq \mathbb{R}$ such that any uncountable subset $B \subseteq A$ is dense in an open interval of $\mathbb{R}$.

Proof .:.

Note that there are only $|\mathbb{R}| = \aleph_1$ nowhere dense sets, being the boundaries of open sets which are themselves countable unions of open intervals. So we can consider the closure of the nowhere dense sets, and enumerate these: $\{N_\alpha : \alpha < \omega_1\} = \{\overline{N} \subseteq \mathbb{R} : N \text{ is nowhere dense}\}$. Just as before, we can then construct the sequence $a_\alpha \in \mathbb{R} \setminus \bigcup_{\beta<\alpha} N_\alpha$, and consider $A = \{a_\alpha : \alpha < \omega_1\}$.

Note that if $B \subseteq A$ is nowhere dense—i.e. is not dense in any open interval of $\mathbb{R}$—then $B = N_\alpha$ for some $\alpha < \omega_1$. But then

$$B = B \cap A = N_\alpha \cap A \subseteq \{a_\beta : \beta \leq \alpha\}$$

is a countable set. ⊣

It might be tempting to do some sort of Cantor set like subset for such a set, but it simply isn't possible under CH. Such a set is really given by the existence of a *luzin* set of reals, a set which only countably intersects every first-category set. Luzin sets, named after Николай Лузин (Nikolai Luzin), exist under CH, like sierpiński sets. And CH turns out to be equivalent to their existence so long as all sets of reals of size $< |\mathbb{R}|$ are first-category.

Section 3. *Harder Results*

3 • 1. Result

(ZFC + CH) There is an uncountable family $\mathcal{F}$ of entire functions (in the sense of differentiable on all of $\mathbb{C}$) such that for each $z \in \mathbb{C}$, the set of values at z is always countable: $|\{f(z) : f \in \mathcal{F}\}| = \aleph_0$.

Proof .:.

Because $|\mathbb{C}| = |\mathbb{R}| \times |\mathbb{R}| = \aleph_1^2 = \aleph_1$ under CH, enumerate $\mathbb{C} = \{c_\alpha : \alpha < \omega_1\}$. Let $\mathbb{Q}(i) = \mathbb{Q} + \mathbb{Q}i$, the rational complex numbers (or whatever they're called). We will define entire functions $\langle f_\alpha : \alpha < \omega_1 \rangle$ such that for $\beta < \omega_1$,

$$\{f_\alpha(c_\beta) : \alpha < \omega_1\} \subseteq \mathbb{Q}(i) \cup \{f_\gamma(c_\beta) : \gamma \leq \beta\}$$

In other words, we will only consider new values at c_β from those already defined, or from $\mathbb{Q}(i)$. Because both $\mathbb{Q}(i)$ and $\{f_\gamma(c_\beta) : \gamma \leq \beta\}$ are countable, this gives the result.

To show this, proceed by recursion. Having dealt with the values of c_β and functions f_β for $\beta < \alpha < \omega_1$, as this is only countably many values, reorder to get the values and functions

$$\{c_\beta : \beta < \alpha\} = \{d_n : n < \omega\} \quad \text{and} \quad \{f_\beta : \beta < \alpha\} = \{g_n : n < \omega\}.$$

The function f_α will be an infinite sum of the form

$$f_\alpha(z) = \varepsilon_0(z - d_0) + \varepsilon_1(z - d_0)(z - d_1) + \cdots$$

for small enough $\langle \varepsilon_n : n \in \omega \rangle$ defined recursively to ensure the series converges. To do this, for each $n < \omega$, suppose $\varepsilon_0, \cdots, \varepsilon_{n-1}$ have been chosen. Choose ε_n such that

$$g_n(d_{n+1}) \neq \varepsilon_0(d_{n+1} - d_0) + \cdots + \varepsilon_n(d_{n+1} - d_0)(d_{n+1} - d_1) \cdots (d_{n+1} - d_n) \in \mathbb{Q}(i),$$

and such that ε_n is small enough. In other words, we are diagonalizing out of the f_βs while remaining in $\mathbb{Q}(i)$ at c_β for $\beta < \alpha$. Once we have the resulting infinite sequence $\langle \varepsilon_n : n \in \omega \rangle$, we can define $f_\alpha(z)$ for all z, not just $z = d_n$ for $n \in \omega$. To show that we can choose ε_n small enough, just let

$$|\varepsilon_n| \cdot n^n (1 + |d_0|) \cdots (1 + |d_n|) < 1/2^n.$$

We then have for $n > |z|$,

$$\begin{aligned} \left|\varepsilon_n(z - d_0) \cdots (z - d_n)\right| &\leq |\varepsilon_n|(n + |d_0|) \cdots (n + |d_n|) \\ &\leq |\varepsilon_n| n^n (1 + |d_0|) \cdots (1 + |d_n|) < 1/2^n. \end{aligned}$$

This means that $f_\alpha(z)$ is defined for every $z \in \mathbb{C}$. This gives the construction. Now that we have f_α for $\alpha < \omega_1$, we can consider whether the property holds of $\mathcal{F} = \{f_\alpha : \alpha < \omega_1\}$.

So let $c_\beta \in \mathbb{C}$ be arbitrary. At later stages $\alpha > \beta$, we defined f_α on all c_γ for $\gamma < \alpha$ to be in $\mathbb{Q}(i)$. Hence $f_\alpha(c_\beta) \in \mathbb{Q}(i)$. Therefore

$$\{f_\alpha(c_\beta) \colon \beta < \alpha < \omega_1\} \subseteq \mathbb{Q}(i).$$

At earlier stages $\alpha \leq \beta$, since $\{f_\gamma(c_\beta) : \gamma \leq \beta\}$ is countable (as $\beta < \omega_1$), there are only countably many possible values of $f_\alpha(c_\beta)$. Hence

$$\{f_\alpha(c_\beta) \colon \alpha < \omega_1\} \subseteq \mathbb{Q}(i) \cup \{f_\alpha(c_\beta) \colon \alpha \leq \beta\}.$$

So there are only countably many possibilities for the value $f_\alpha(c_\beta)$ for $f_\alpha \in \mathcal{F}$, and thus the result holds. ⊣

This marks the last proof of this document, but I think it's interesting to at least list a some more consequences of CH that can't really be proven in this kind of time frame. The proofs aren't hard to find, however, with proofs of the next two in [3], a proof of Result 3 • 5 in [5], and an overview of Kaplansky's Conjecture (3 • 4)—with references to the actual proofs—found in [1].

3 • 2. Result

(ZFC + CH) There is a function $f : \mathbb{R} \to \mathbb{R}$ such that for all $x \in \mathbb{R}$,

$$\lim_{h\to 0} \max(f(x-h), f(x+h)) = \infty.$$

3 • 3. Result

(ZFC + CH) There is a surjection $F : \mathbb{R} \to \mathbb{R}^2$ of the form $F(x) = \langle f(x), g(x)\rangle$ such that for all $x \in \mathbb{R}$, either f or g is differentiable at x.

This next result is one that helped show the independence of Kaplansky's conjecture (on banach algebras) from ZFC. In particular, CH disproves the conjecture.

3 • 4. Result (Kaplansky's Conjecture)

Let X be an infinite, compact Hausdorff space with $C(X)$ the space of continuous functions from X to $\mathbb{C}$.
If CH holds, then there is an incomplete norm on $C(X)$. In other words, CH implies there is a norm non-equivalent to the $\| \|_\infty$ norm on $C(X)$.

Note that all complete norms on $C(X)$ are equivalent to the $\| \|_\infty$ norm.

3 • 5. Result

Let $\mathbb{E}_n$ be a field for $n \in \omega$. If CH holds, the global dimension of the direct product $\prod_{n<\omega} \mathbb{E}_n$ is 2.

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